Response of non-equilibrium systems at criticality: exact results for the Glauber-Ising chain

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Abstract. We investigate the non-equilibrium two-time correlation and response functions and the associated fluctuation–dissipation ratio for the ferromagnetic Ising chain with Glauber dynamics. The scaling behaviour of these quantities at low temperature and large times is studied in detail. This analysis encompasses the self-similar domain-growth (aging) regime, the spatial and temporal Porod regimes, and the convergence toward equilibrium. The fluctuation–dissipation ratio admits a non-trivial limit value $X_\infty = \frac{1}{2}$ at zero temperature, and more generally in the aging regime.

1. Introduction

The ferromagnetic Ising chain with Glauber dynamics [1] is one of the simplest non-equilibrium systems. Consider a finite system consisting of N Ising spins $\sigma_n = \pm 1$, with Hamiltonian

$$\mathcal{H} = -J \sum_n \sigma_n \sigma_{n+1}. \tag{1.1}$$

In its heat-bath formulation the use of Glauber dynamics consists of picking, at every time step $\delta t = 1/N$, a site $n = 1, \dots, N$ at random, and updating its spin $\sigma_n(t)$ according to the stochastic rule

$$\sigma_n(t) \rightarrow \begin{cases} +1 & \text{with prob. } \dfrac{1 + \tanh(\beta h_n(t))}{2} \\ -1 & \text{with prob. } \dfrac{1 - \tanh(\beta h_n(t))}{2} \end{cases} \tag{1.2}$$

where $\beta = 1/T$ is the inverse temperature, and the local field $h_n(t)$ acting on $\sigma_n(t)$ reads

$$h_n(t) = J(\sigma_{n-1}(t) + \sigma_{n+1}(t)). \tag{1.3}$$

In the thermodynamic limit ($N \rightarrow \infty$), each site is thus updated according to a Poisson process.

At positive temperature, starting from a random initial condition, obtained, for instance, by a quench from infinitely high temperature, the system relaxes to its paramagnetic equilibrium state. The situation at zero temperature is different. Indeed, still starting from a random initial condition, the system is unable to relax to any of its two ferromagnetically ordered, symmetry-related, equilibrium states. In contrast, domains of positive and negative magnetization grow forever, and, in the scaling regime, the system becomes statistically self-similar with only one

characteristic length scale, the mean size of domains. This coarsening process is a consequence of the existence of spontaneous symmetry breaking [2, 3].

The Ising chain is special because its critical temperature T_c is equal to zero. However, its dynamical behaviour illustrates that of generic statistical-mechanical models in the absence of quenched disorder. For the latter, starting from a random initial condition, the system relaxes exponentially to equilibrium in the high-temperature phase ($T > T_c$). At equilibrium, two-time quantities such as the correlation function $C(t, s)$ or the response function $R(t, s)$, where s (waiting time) is smaller than t (observation time), only depend on the time difference

$$\tau = t - s \geqslant 0 \tag{1.4}$$

and they are simply related to each other by the fluctuation–dissipation theorem:

$$R_{eq}(\tau) = -\beta \frac{dC_{eq}(\tau)}{d\tau}. \tag{1.5}$$

In the low-temperature phase ($T < T_c$) the system undergoes phase ordering. In this non-equilibrium situation, $C(t, s)$ and $R(t, s)$ are non-trivial functions of both time variables, which only depend on their ratio at late times, i.e., in the self-similar scaling regime. This behaviour is usually referred to as aging (for recent review, see [4]). Moreover, no such simple relation as equation (1.5) holds between correlation and response, i.e., $R(t, s)$ and $\partial C(t, s)/\partial s$ are no longer proportional. It is then natural to characterize the distance to equilibrium of an aging system by the so-called fluctuation–dissipation ratio [4–6]

$$X(t, s) = \frac{T R(t, s)}{\partial C(t, s)/\partial s}. \tag{1.6}$$

In recent years, several works [4–9] have been devoted to the study of the fluctuation–dissipation ratio. In the low-temperature phase of aging systems, such as glasses and spin glasses, or of systems exhibiting domain growth, $X(t, s)$ turns out to be a non-trivial function of its two arguments. In the case of coarsening systems, analytical and numerical studies indicate that the limit fluctuation–dissipation ratio,

$$X_\infty = \lim_{s \to \infty} \lim_{t \to \infty} X(t, s) \tag{1.7}$$

vanishes throughout the low-temperature phase [7, 8].

However, to date, very little attention has been devoted to the fluctuation–dissipation ratio $X(t, s)$ for non-equilibrium systems *at criticality*. From now on, we will only consider non-disordered systems. For instance, one may wonder whether there exists, for a given model, a well-defined limit X_∞ at $T = T_c$, different from its value in the low-temperature phase. Indeed, *a priori*, for a system without disorder, such as a ferromagnet, quenched from infinitely high temperature to its critical point, the limit fluctuation–dissipation ratio X_∞ at $T = T_c$ (if it exists) may take any value between $X_\infty = 1$ ($T > T_c$: equilibrium) and $X_\infty = 0$ ($T < T_c$: domain growth).

The only cases of critical systems for which the fluctuation–dissipation ratio has been considered are, to our knowledge, the models of [5] (the random walk model, the free Gaussian field model, and the two-dimensional X–Y model at zero temperature) which share the limit fluctuation–dissipation ratio $X_\infty = \frac{1}{2}$, and the backgammon model, a mean-field model for which $T_c = 0$, where it has been shown that $X_\infty = 1$, up to a large logarithmic correction, for both energy and density fluctuations [10, 11].

In this work we investigate the non-equilibrium response function and fluctuation–dissipation ratio for the Glauber–Ising chain, both at zero temperature and in the scaling regime of low temperatures. Exact results for these two-time quantities are derived, exploiting the solvability of the model. We then perform a detailed analysis of their scaling behaviour

at low temperature and large times. Since the computation of $R(t, s)$ and $X(t, s)$ requires the knowledge of the equal-time correlation function, the two-time correlation $C(t, s)$, and of its derivative with respect to one of the two times (see equation (1.6)), we will include the computation and the scaling analysis of the latter quantities, though a number of papers have already dealt with the study of correlations in the Glauber–Ising chain [1, 12–16]. Our presentation will thus be complete and self-contained, and the computation of all quantities will follow the same integral-transform methods.

In a forthcoming publication [17], we will present a study of the critical response and fluctuation–dissipation ratio for the ferromagnetic spherical model in any dimension $d > 2$, and for the two-dimensional Ising model. Both models possess a whole low-temperature phase for $T < T_c$, hence they are faithful representatives of generic domain-growth systems.

One salient outcome of these joint works is the realization that the limit fluctuation–dissipation ratio X_∞ is a novel universal characteristic of critical dynamics (see the discussion at the end of this paper). For the Ising chain at its critical temperature $T_c = 0$, we obtain the value $X_\infty = \frac{1}{2}$. The occurrence of a common value for X_∞ between the Ising chain and the models considered in [5] seems rather coincidental, for the time being.

In the following, we shall make use of the parameters μ and γ, defined by

$$\mu = -\ln\tanh(\beta J) \qquad \gamma = \tanh(2\beta J) = \frac{1}{\cosh\mu}. \tag{1.8}$$

These parameters are related to the equilibrium correlation length ξ_{eq} [18] and the equilibrium relaxation time τ_{eq} (see equation (2.16)) by

$$\xi_{eq} = \frac{1}{\mu} \qquad \tau_{eq} = \frac{1}{1-\gamma}. \tag{1.9}$$

Both diverge exponentially quickly as the critical temperature $T_c = 0$ is approached, according to

$$\xi_{eq} \approx \frac{e^{2\beta J}}{2} \qquad \tau_{eq} \approx \frac{2}{\mu^2} \approx \frac{e^{4\beta J}}{2}. \tag{1.10}$$

Hence we have the scaling law

$$\tau_{eq} \approx 2\xi_{eq}^2 \tag{1.11}$$

corresponding to a dynamical critical exponent $z = 2$, and reflecting the diffusive nature of domain growth at zero temperature.

We close this introduction with a discussion about relevant timescales. For two-time quantities such as $C(t, s)$, $R(t, s)$, $X(t, s)$, three timescales are to be compared in the scaling regime of long times and low temperature, namely s, $\tau = t - s$ and τ_{eq}. Considering any two of these timescales as comparable, with an arbitrary ratio between them, and small (or large) compared to the third one, six different regimes can be defined *a priori*. Three of them will be of interest in this work, which we summarize here for convenience:

$$\tau \sim \tau_{eq} \ll s : \qquad \text{equilibrium} \tag{1.12}$$

$$s \sim \tau \ll \tau_{eq} : \qquad \text{self-similar domain growth (aging)} \tag{1.13}$$

$$\tau \ll s \sim \tau_{eq} : \qquad \text{early-time or temporal Porod regime.} \tag{1.14}$$

Aging persists forever at zero temperature, just as the coarsening process itself, while it is interrupted at the timescale τ_{eq} at any low but finite temperature.

2. Magnetization profile

In this section we present the methods used throughout the paper applied to the example of the relaxation of the magnetization profile

$$M_n(t) = \langle \sigma_n(t) \rangle \tag{2.1}$$

where the brackets denote an average over the thermal history of the system and over the initial conditions, which are unspecified for the time being.

The time evolution of the magnetization $M_n(t)$ is readily deduced from the stochastic rule (1.2), and is given by

$$\frac{dM_n(t)}{dt} = -M_n(t) + \langle \tanh(\beta h_n(t)) \rangle. \tag{2.2}$$

Since the local field h_n only assumes three symmetric values, 0 and $\pm 2J$, we have

$$\tanh(\beta h_n) = \frac{\gamma}{2J} h_n = \frac{\gamma}{2}(\sigma_{n-1} + \sigma_{n+1}) \tag{2.3}$$

leading to the linear evolution equations

$$\frac{dM_n(t)}{dt} = -M_n(t) + \frac{\gamma}{2}(M_{n-1}(t) + M_{n+1}(t)). \tag{2.4}$$

The existence of closed, linear equations for the magnetization, and more generally for higher-order correlation functions, ensures the solvability of the model [1].

In order to solve coupled equations of the form (2.4), we shall make extensive use of Laplace and Fourier transforms. For any quantity $f_n(t)$, depending both on continuous time t and on the discrete site label n, we introduce:

- the temporal Laplace transform

$$f_n^L(p) = \int_0^\infty f_n(t)e^{-pt} \, dt \qquad f_n(t) = \int \frac{dp}{2\pi i} f_n^L(p)e^{pt} \tag{2.5}$$

- the spatial Fourier transform

$$f^F(q, t) = \sum_n f_n(t)e^{-inq} \qquad f_n(t) = \int_0^{2\pi} \frac{dq}{2\pi} f^F(q, t)e^{inq} \tag{2.6}$$

- the Fourier–Laplace transform

$$f^{FL}(q, p) = \sum_n \int_0^\infty f_n(t)e^{-(pt+inq)} \, dt \qquad f_n(t) = \int \frac{dp}{2\pi i} \int_0^{2\pi} \frac{dq}{2\pi} f^{FL}(q, p)e^{pt+inq}. \tag{2.7}$$

Using the above integral transforms, equation (2.4) can be recast as

$$\frac{dM^F(q, t)}{dt} = (\gamma \cos q - 1)M^F(q, t) \tag{2.8}$$

or else as

$$pM^{FL}(q, p) = (\gamma \cos q - 1)M^{FL}(q, p) + M^F(q, t = 0) \tag{2.9}$$

yielding

$$M^{FL}(q, p) = \frac{M^F(q, t = 0)}{p + 1 - \gamma \cos q}. \tag{2.10}$$

Consider first the locally magnetized initial condition where the spin at the origin is pointing upwards, i.e., $\sigma_0(t = 0) = +1$, but the configuration is otherwise totally random. The

corresponding solution $G_n(t)$ to equation (2.4) is the Green function of the problem. We have $G_n(t=0) = \delta_{n,0}$, so that $G^F(q, t=0) = 1$. Equation (2.10) then reads

$$G^{FL}(q, p) = \frac{1}{p + 1 - \gamma \cos q} \tag{2.11}$$

hence

$$G^F(q, t) = e^{(\gamma \cos q - 1)t} \tag{2.12}$$

and

$$G_n(t) = e^{-t} \int_0^{2\pi} \frac{dq}{2\pi} e^{\gamma t \cos q + inq} = e^{-t} I_n(\gamma t) \tag{2.13}$$

where I_n is the modified Bessel function. Equation (2.11) also yields

$$G_n^L(p) = \frac{1}{\sqrt{(p+1)^2 - \gamma^2}} \left(\frac{p + 1 - \sqrt{(p+1)^2 - \gamma^2}}{\gamma} \right)^{|n|}. \tag{2.14}$$

Returning to equation (2.4), its general solution in direct space is obtained by inverting equation (2.10), yielding the following spatial convolution:

$$M_n(t) = M_n(t=0) * G_n(t) = \sum_m M_m(t=0) G_{n-m}(t). \tag{2.15}$$

In particular, for a uniformly magnetized system, i.e., $M_n(t=0) = M$ for all n, we have an exact exponential relaxation

$$\frac{M_n(t)}{M} = \sum_n G_n(t) = G^F(q=0, t) = e^{-t/\tau_{eq}} \tag{2.16}$$

at any finite temperature, where the relaxation time τ_{eq} is given by equation (1.9).

Throughout the following, we shall be mostly interested in the non-equilibrium scaling regime of long times, large distances and low temperatures, such that t and τ_{eq} are simultaneously large but comparable, i.e., their ratio t/τ_{eq} is arbitrary, and the same holds for n and ξ_{eq}:

$$t \sim \tau_{eq} \gg 1 \qquad n \sim \xi_{eq} \gg 1. \tag{2.17}$$

The scaling law (1.11) then implies

$$t \sim \tau_{eq} \sim n^2 \sim \xi_{eq}^2. \tag{2.18}$$

As a consequence, $p \sim q^2 \sim \mu^2$ are simultaneously small, and equation (2.11) simplifies to

$$G^{FL}(q, p) \approx \frac{2}{2p + q^2 + \mu^2}. \tag{2.19}$$

Performing the inverse Laplace transform first, we obtain

$$G_n(t) \approx \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{t}{\tau_{eq}} - \frac{n^2}{2t} \right) \tag{2.20}$$

which is the scaling form of the Green function (2.13), involving the variables t/τ_{eq} and $n/\sqrt{t}$, in agreement with equation (2.18). At zero temperature the exponential damping factor $\exp(-t/\tau_{eq})$ is absent, so that $G_n(t)$ is a function of $n/\sqrt{t}$ only, reflecting the statistical self-similarity of the coarsening process. In this case the Gaussian profile is normalized, again emphasizing the underlying diffusive mechanism.

3. Equal-time correlation function

The equal-time correlation function between any two spins is defined as

$$C(m, n, t) = \langle \sigma_m(t)\sigma_n(t)\rangle. \tag{3.1}$$

In the following, unless otherwise specified, we shall consider a random initial condition, defined by averaging over all possible initial configurations with equal weights. This procedure corresponds to quenching the system at time $t = 0$ from its equilibrium state at infinitely high temperature. Because the invariance under spatial translations along the chain is preserved by the dynamics, the correlation function $C(m, n, t)$ only depends on the distance $|n - m|$ between both spins. We denote it as

$$C(m, n, t) = C_{n-m}(t). \tag{3.2}$$

We have, in particular,

$$C_0(t) = 1. \tag{3.3}$$

The equal-time correlation function can be shown, in analogy with equation (2.4), to obey coupled linear differential equations of the form [1]

$$\frac{\mathrm{d}C_n(t)}{\mathrm{d}t} = -2C_n(t) + \gamma(C_{n-1}(t) + C_{n+1}(t)) \qquad (n \neq 0) \tag{3.4}$$

with condition (3.3), and the initial value $C_n(t = 0) = \delta_{n,0}$.

In order to solve equation (3.4), we complete them by the corresponding equation for $n = 0$, with a time-dependent source $v(t)$ in the right-hand side, to be determined in such a way that condition (3.3) be fulfilled. In other words, we consider the equations

$$\frac{\mathrm{d}C_n(t)}{\mathrm{d}t} = -2C_n(t) + \gamma(C_{n-1}(t) + C_{n+1}(t)) + v(t)\delta_{n,0} \tag{3.5}$$

which read, in Fourier–Laplace space,

$$pC^{\mathrm{FL}}(q, p) = 2(\gamma \cos q - 1)C^{\mathrm{FL}}(q, p) + v^{\mathrm{L}}(p) + 1 \tag{3.6}$$

yielding

$$C^{\mathrm{FL}}(q, p) = \frac{v^{\mathrm{L}}(p) + 1}{p + 2 - 2\gamma \cos q}. \tag{3.7}$$

Condition (3.3) then reads

$$\int_0^{2\pi} \frac{\mathrm{d}q}{2\pi} C^{\mathrm{FL}}(q, p) = \frac{v^{\mathrm{L}}(p) + 1}{\sqrt{(p + 2)^2 - 4\gamma^2}} = \frac{1}{p} \tag{3.8}$$

hence

$$v^{\mathrm{L}}(p) = \frac{\sqrt{(p + 2)^2 - 4\gamma^2}}{p} - 1. \tag{3.9}$$

We thus obtain the result

$$C^{\mathrm{FL}}(q, p) = \frac{\sqrt{(p + 2)^2 - 4\gamma^2}}{p(p + 2 - 2\gamma \cos q)} \tag{3.10}$$

or else

$$C_n^{\mathrm{L}}(p) = \frac{1}{p}\left(\frac{p + 2 - \sqrt{(p + 2)^2 - 4\gamma^2}}{2\gamma}\right)^{|n|} \tag{3.11}$$

which we now discuss.

Equilibrium. At any finite temperature, and for long enough times ($t \gg \tau_{eq}$), the correlation function $C_n(t)$ converges to its equilibrium value, $C_{n,eq} = \lim_{p\to 0}(pC_n^L(p))$. From equation (3.11) we recover the well known expression [18]

$$C_{n,eq} = \left(\frac{1 - \sqrt{1 - \gamma^2}}{\gamma}\right)^{|n|} = e^{-\mu|n|}. \tag{3.12}$$

Non-equilibrium. Let us analyse the non-equilibrium properties in the regime (2.18) of low temperatures and long times. In this regime expressions (3.10) and (3.11) scale as

$$C^{FL}(q, p) \approx \frac{2\sqrt{p + \mu^2}}{p(p + q^2 + \mu^2)} \tag{3.13}$$

and

$$C_n^L(p) \approx \frac{e^{-|n|\sqrt{p+\mu^2}}}{p}. \tag{3.14}$$

Equation (A.1) leads to the following explicit scaling form:

$$C_n(t) \approx \frac{1}{2}\left\{e^{\mu|n|}\,\mathrm{erfc}\left(\frac{|n|}{2\sqrt{t}} + \mu\sqrt{t}\right) + e^{-\mu|n|}\,\mathrm{erfc}\left(\frac{|n|}{2\sqrt{t}} - \mu\sqrt{t}\right)\right\} \tag{3.15}$$

for the equal-time correlation function, involving two scaling variables, $\mu^2 t \approx 2t/\tau_{eq}$ and $n/\sqrt{t}$. The error function, erf z, and the complementary error function, erfc z, are defined as

$$\mathrm{erf}\, z = \frac{2}{\sqrt{\pi}}\int_0^z e^{-x^2}\,\mathrm{d}x \qquad \mathrm{erfc}\, z = 1 - \mathrm{erf}\, z = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-x^2}\,\mathrm{d}x. \tag{3.16}$$

An alternative expression for $C_n(t)$ is

$$C_n(t) \approx \frac{|n|}{2}\int_0^t \frac{\mathrm{d}u}{\sqrt{\pi u^3}}\exp\left(-\mu^2 u - \frac{n^2}{4u}\right) \tag{3.17}$$

which can be obtained from either equation (3.13) or (3.15). The following limiting situations are of interest.

(i) At zero temperature, and more generally in the domain-growth regime ($1 \ll t \ll \tau_{eq}$), we can set $\mu = 0$ in equation (3.15), which thus simplifies to

$$C_n(t) \approx \mathrm{erfc}\left(\frac{|n|}{2\sqrt{t}}\right). \tag{3.18}$$

This expression [3, 15] involves only one scaling variable, $n/\sqrt{t}$, reflecting the fact that, in this regime, the pattern formed by the domains is self-similar.

(ii) In the opposite regime ($t \gg \tau_{eq}$), expression (3.15) converges exponentially quickly to its equilibrium value (3.12), as

$$C_n(t) - C_{n,eq} \approx -\frac{\mu|n|}{4\sqrt{2\pi}}\left(\frac{t}{\tau_{eq}}\right)^{-3/2}e^{-2t/\tau_{eq}}. \tag{3.19}$$

(iii) The short-distance or Porod regime

$$1 \ll |n| \ll \xi_{eq} \sim \sqrt{t} \tag{3.20}$$

is yet another situation of interest [3]. In this regime, equation (3.15) becomes

$$C_n(t) \approx 1 - A(t)|n| \tag{3.21}$$

with

$$A(t) = \frac{e^{-\mu^2 t}}{\sqrt{\pi t}} + \mu \, \mathrm{erf}(\mu\sqrt{t}). \tag{3.22}$$

This result may also be found from equation (3.14), by noting that $|n|\sqrt{p + \mu^2} \ll 1$, and using equation (A.2). Expression (3.22) for the amplitude $A(t)$ interpolates between the power-law behaviour

$$A(t) \approx \frac{1}{\sqrt{\pi t}} \tag{3.23}$$

in the domain-growth regime, in agreement with equation (3.18), and the equilibrium value

$$A_{\mathrm{eq}} \approx \mu \tag{3.24}$$

in agreement with equation (3.12).

Equation (3.21) also holds for finite values $|n| = 0, 1, 2, \ldots$ of the distance between spins, in the scaling regime ($\xi_{\mathrm{eq}} \gg 1$, $t \gg 1$). In particular, the density of defects (domain walls) in the system reads

$$\rho_{\mathrm{def}}(t) = \frac{1 - C_1(t)}{2} \approx \frac{A(t)}{2}. \tag{3.25}$$

The result (3.21) is thus in quantitative agreement with the general predictions given in [3] on the so-called Porod singularities of correlation functions, whose form depends on the nature of topological defects, and whose amplitude is known in terms of the density ρ_{def} of these defects.

The spatial range over which ferromagnetic order has propagated at time t can be measured by the dimensionless susceptibility

$$\chi(t) = \sum_n C_n(t) = C^{\mathrm{F}}(q = 0, t) \tag{3.26}$$

for which equation (3.10) yields

$$\chi^{\mathrm{L}}(p) = \frac{1}{p}\sqrt{\frac{p + 2 + 2\gamma}{p + 2 - 2\gamma}}. \tag{3.27}$$

Throughout the non-equilibrium scaling regime (2.18), we have

$$\chi^{\mathrm{L}}(p) \approx \frac{2}{p\sqrt{p + \mu^2}} \tag{3.28}$$

i.e., using equation (A.3),

$$\chi(t) \approx \frac{2}{\mu} \, \mathrm{erf}(\mu\sqrt{t}). \tag{3.29}$$

This result interpolates between the square-root behaviour

$$\chi(t) \approx 4\sqrt{\frac{t}{\pi}} \tag{3.30}$$

in the domain-growth regime ($t \ll \tau_{\mathrm{eq}}$), in agreement with equation (3.18), and the limit

$$\chi_{\mathrm{eq}} \approx \frac{2}{\mu} \tag{3.31}$$

as equilibrium is approached ($t \gg \tau_{\mathrm{eq}}$). This last expression is the scaling behaviour of the exact equilibrium value

$$\chi_{\mathrm{eq}} = \operatorname{cotanh} \frac{\mu}{2} = \frac{\gamma + 1 - \sqrt{1 - \gamma^2}}{\gamma - 1 + \sqrt{1 - \gamma^2}} = e^{2\beta J} \tag{3.32}$$

corresponding to equation (3.12).

Finally, the dimensionless product

$$Q(t) = A(t)\chi(t) = 2\operatorname{erf}(\mu\sqrt{t})\left(\frac{e^{-\mu^2 t}}{\mu\sqrt{\pi t}} + \operatorname{erf}(\mu\sqrt{t})\right) \tag{3.33}$$

is a form factor characteristic of the correlation profile. It increases from the value $Q = 4/\pi$ in the domain-growth (aging) regime (see equations (3.23), (3.30)) to the equilibrium value $Q_{\mathrm{eq}} = 2$ (see equations (3.24), (3.31)).

4. Two-time correlation function

We now consider the two-time correlation function

$$C(m, n, t, s) = \langle \sigma_n(t)\sigma_m(s) \rangle \tag{4.1}$$

where s is the waiting time and $t = s + \tau$ is the observation time. For a random initial condition, invariance under spatial translations yields

$$C(m, n, t, s) = C_{n-m}(t, s). \tag{4.2}$$

This two-time correlation function can be shown, again in analogy with equation (2.4), to obey the coupled linear partial differential equations [1]

$$\frac{\partial C_n(t, s)}{\partial t} = -C_n(t, s) + \frac{\gamma}{2}(C_{n-1}(t, s) + C_{n+1}(t, s)) \tag{4.3}$$

for $t > s$, with the initial value

$$C_n(s, s) = C_n(s) \tag{4.4}$$

at time $t = s$, i.e., $\tau = 0$, where the right-hand side is the equal-time correlation function, given in Fourier–Laplace space by equations (3.10) and (3.11).

The second argument s plays the role of a parameter in equation (4.3), hence this equation is formally identical to equation (2.4), with the initial condition (4.4) playing the role of $M_n(t = 0)$. The solution of equation (4.3), seen as an evolution equation in the τ variable, is, therefore, formally identical to that of equation (2.4), and reads

$$C_n(s + \tau, s) = C_n(s) * G_n(\tau) = \sum_m C_m(s)G_{n-m}(\tau). \tag{4.5}$$

In order to write down this solution explicitly, it is convenient to introduce the double-Laplace transform of the function $C_n(s + \tau, s)$, where p_s is conjugate to s, the transform being denoted by L_s, and p is conjugate to τ, the transform being denoted by L:

$$C_n^{\mathrm{LL}_s}(p, p_s) = \int_0^\infty \int_0^\infty C_n(s + \tau, s)e^{-(p_s s + p\tau)} \, \mathrm{d}s \, \mathrm{d}\tau. \tag{4.6}$$

With this definition, the solution of equation (4.3) reads

$$C^{\mathrm{FLL}_s}(q, p, p_s) = \frac{C^{\mathrm{FL}}(q, p_s)}{p + 1 - \gamma \cos q} \tag{4.7}$$

i.e., using equation (3.10),

$$C^{\text{FLL}_s}(q, p, p_s) = \frac{\sqrt{(p_s + 2)^2 - 4\gamma^2}}{p_s(p_s + 2 - 2\gamma \cos q)(p + 1 - \gamma \cos q)}. \tag{4.8}$$

In the following, we shall mostly be interested in the diagonal component of the correlation function, or autocorrelation function, $C_0(t, s)$. Equation (4.8) yields, upon integration over q,

$$C_0^{\text{LL}_s}(p, p_s) = \frac{1}{p_s(p_s - 2p)} \left(\sqrt{\frac{(p_s + 2)^2 - 4\gamma^2}{(p + 1)^2 - \gamma^2}} - 2 \right). \tag{4.9}$$

The discussion given at the end of section 3 on the behaviour of $C_n(t)$ in various regimes is readily extended to the two-time correlation function $C_n(t, s)$ considered in the present section, using its integral-transform expressions (4.8), (4.9).

Equilibrium.   The equilibrium correlation function is obtained by letting $s \to \infty$ (in practice, $s \gg \tau_{\text{eq}}$), while keeping $\tau = t - s$ fixed. Equation (4.8) simplifies to

$$C_{\text{eq}}^{\text{FL}}(q, p) = \frac{\sqrt{1 - \gamma^2}}{(1 - \gamma \cos q)(p + 1 - \gamma \cos q)} \tag{4.10}$$

which yields, performing the inverse Laplace transform first,

$$C_{n,\text{eq}}(\tau) = e^{-\mu|n|} - \sqrt{1 - \gamma^2} \int_0^\tau G_n(u) \, du = \sqrt{1 - \gamma^2} \int_\tau^\infty G_n(u) \, du. \tag{4.11}$$

In particular, the equilibrium autocorrelation function reads

$$C_{0,\text{eq}}(\tau) = \sqrt{1 - \gamma^2} \int_\tau^\infty G_0(u) \, du. \tag{4.12}$$

In the scaling regime, where τ and τ_{eq} are large and comparable, using equation (2.20), the above formula simplifies to

$$C_{0,\text{eq}}(\tau) \approx \text{erfc} \left(\sqrt{\frac{\tau}{\tau_{\text{eq}}}} \right). \tag{4.13}$$

Non-equilibrium.   The generalization of the non-equilibrium scaling regime (2.18) of long times, large distances, and low temperatures, to the present case of two temporal variables consists of taking s, τ, and τ_{eq} simultaneously large but comparable, with arbitrary ratios between them, and the same for n and ξ_{eq}, i.e., using equation (1.11) again,

$$s \sim \tau \sim \tau_{\text{eq}} \sim n^2 \sim \xi_{\text{eq}}^2 \gg 1. \tag{4.14}$$

As a consequence, $p_s \sim p \sim q^2 \sim \mu^2$ are simultaneously small. In this regime, equation (4.9) reads

$$C_0^{\text{LL}_s}(p, p_s) \approx \frac{2}{p_s \sqrt{2p + \mu^2}(\sqrt{p_s + \mu^2} + \sqrt{2p + \mu^2})}. \tag{4.15}$$

Performing first the inverse Laplace transform of this expression with respect to p, using equation (A.4), we get

$$C_0^{\text{L}_s}(\tau, p_s) \approx \frac{e^{p_s \tau/2}}{p_s} \, \text{erfc} \left(\sqrt{(p_s + \mu^2)\tau/2} \right). \tag{4.16}$$

Then performing the inverse Laplace transform with respect to p_s, using equation (A.5), we obtain

$$C_0(s+\tau, s) \approx \frac{\sqrt{2\tau}}{\pi} e^{-\mu^2\tau/2} \int_0^s \frac{e^{-\mu^2 u}}{(2u+\tau)\sqrt{u}} \, du. \tag{4.17}$$

The change of variable $u = (\tau/2) \tan^2 \theta$ yields

$$C_0(s+\tau, s) \approx \frac{2}{\pi} \int_0^\Theta e^{-\mu^2\tau/(2\cos^2\theta)} \, d\theta \tag{4.18}$$

with

$$\Theta = \arctan\sqrt{\frac{2s}{\tau}}. \tag{4.19}$$

Equation (4.18) gives the general scaling form of the two-time autocorrelation function. It can be alternatively expressed as a double-Gaussian integral

$$C_0(s+\tau, s) \approx \frac{4}{\pi} \int_{\mu\sqrt{\tau/2}}^\infty d\xi \int_0^{\xi\sqrt{2s/\tau}} e^{-(\xi^2+\eta^2)} \, d\eta. \tag{4.20}$$

Only in the symmetric situation $\Theta = \pi/4$, i.e., $\tau = 2s$, or $t = 3s$, does this integral simplify to

$$C_0(3s, s) \approx \tfrac{1}{2}(1 - \operatorname{erf}^2(\mu\sqrt{s})). \tag{4.21}$$

Yet another reformulation of the result (4.18) reads

$$C_0(s+\tau, s) \approx \frac{2}{\pi}\sqrt{\frac{2s}{\tau}} e^{-\mu^2\tau/2} \Phi_1\left(\frac{1}{2}, 1, \frac{3}{2}; -\frac{2s}{\tau}, -\mu^2 s\right) \tag{4.22}$$

where Φ_1 is the confluent hypergeometric series in two variables [19], namely the scaling variables $2s/\tau$ and $\mu^2 s \approx 2s/\tau_{\text{eq}}$.

The fluctuation–dissipation ratio (1.6) involves the derivative of $C_0(t, s)$ with respect to s at fixed observation time t, for which equation (4.18) yields the simpler expression

$$\frac{\partial C_0(t=s+\tau, s)}{\partial s} \approx \left(\frac{2(s+\tau)}{2s+\tau} \frac{e^{-\mu^2 s}}{\sqrt{\pi s}} + \mu \operatorname{erf}(\mu\sqrt{s})\right) \frac{e^{-\mu^2\tau/2}}{\sqrt{2\pi\tau}}. \tag{4.23}$$

Let us discuss some limiting cases of equations (4.18) and (4.23).

(i) At zero temperature, and more generally in the domain-growth regime (1.13), the autocorrelation function exhibits aging, i.e., it only depends on the ratio t/s. Equation (4.18) indeed simplifies for $\mu = 0$ to $C_0(t, s) \approx 2\Theta/\pi$, i.e.,

$$C_0(s+\tau, s) \approx \frac{2}{\pi} \arctan\sqrt{\frac{2s}{\tau}}. \tag{4.24}$$

This expression [13–16] assumes the scaling form

$$C_0(s+\tau, s) \approx F(x) \tag{4.25}$$

with

$$x = \frac{t}{s} = 1 + \frac{\tau}{s} \geqslant 1 \tag{4.26}$$

and

$$F(x) = \frac{2}{\pi} \arctan\sqrt{\frac{2}{x-1}}. \tag{4.27}$$

We have $F(3) = \frac{1}{2}$, in agreement with equation (4.21). Similarly, equation (4.23) becomes

$$\frac{\partial C_0(t, s)}{\partial s} \approx \frac{F_1(x)}{s} \tag{4.28}$$

with $F_1(x) = -x \, dF(x)/dx$, i.e.,

$$F_1(x) = \frac{x}{\pi(x+1)} \sqrt{\frac{2}{x-1}}. \tag{4.29}$$

(ii) For $s \to \infty$ with τ fixed, i.e., $p_s \to 0$, equation (4.18) converges to the equilibrium form (4.13); this is obvious using equation (4.16).

(iii) The early-time regime, or temporal Porod regime (1.14), is the counterpart of the spatial Porod regime (3.20). From equation (4.23), we find

$$\frac{\partial C_0(s + \tau, s)}{\partial s} \approx \frac{A(s)}{\sqrt{2\pi\tau}} \tag{4.30}$$

where $A(s)$ has been defined in equation (3.22). We thus obtain, to leading order,

$$C_0(s + \tau, s) \approx 1 - A(s)\sqrt{\frac{2\tau}{\pi}}. \tag{4.31}$$

The replacement of $|n|$ in equation (3.21) by (a constant times) $\sqrt{\tau}$ in equation (4.31) reflects, once more, the underlying diffusive mechanism. Finally, the behaviour (3.23) of $A(s)$ in the domain-growth regime is in agreement with equation (4.24), while equation (3.24) matches (4.13). The generalization to the spatio-temporal Porod regime

$$1 \ll n^2 \sim \tau \ll s \sim \tau_{\text{eq}} \tag{4.32}$$

is straightforward. One has, using equations (4.5), (3.21) and (2.20),

$$C_n(s + \tau, s) \approx (1 - A(s)|n|) * \frac{e^{-n^2/(2\tau)}}{\sqrt{2\pi\tau}} \tag{4.33}$$

where the convolution involves the spatial coordinate n. Replacing the discrete sum by an integral, we obtain

$$C_n(s + \tau, s) \approx 1 - A(s) \left\{ \sqrt{\frac{2\tau}{\pi}} e^{-n^2/(2\tau)} + n \operatorname{erf}\left(\frac{n}{\sqrt{2\tau}}\right) \right\}. \tag{4.34}$$

This result interpolates between the spatial behaviour (3.21) of the equal-time correlation function, for $\tau = 0$, and the temporal behaviour (4.31) of the autocorrelation function, for $n = 0$.

5. Two-time response function

Suppose now that the system is subjected to a small magnetic field $H_n(t)$, depending on the site label n and on time $t > 0$ in an arbitrary fashion. This amounts to adding to the ferromagnetic Hamiltonian (1.1) a time-dependent perturbation of the form

$$\delta\mathcal{H}(t) = -\sum_n H_n(t)\sigma_n(t). \tag{5.1}$$

The dynamics of the model is still given by the stochastic rule (1.2), where the local field $h_n(t)$ acting on the spin $\sigma_n(t)$ now reads

$$h_n(t) = J(\sigma_{n-1}(t) + \sigma_{n+1}(t)) + H_n(t). \tag{5.2}$$

We again consider a random initial condition. Causality and invariance under spatial translations imply that the magnetization $M_n(t)$ at time t reads

$$M_n(t) = \langle \sigma_n(t) \rangle = \beta \int_0^t \mathrm{d}u \sum_m R_{n-m}(t, u) H_m(u) + \cdots \tag{5.3}$$

to first order in the magnetic fields $H_n(t)$. This formula defines the two-time dimensionless response function $R_{n-m}(t, s)$ of the model. A more formal definition reads

$$R_{n-m}(t, s) = T \left. \frac{\delta M_n(t)}{\delta H_m(s)} \right|_{\{H=0\}}. \tag{5.4}$$

The evolution equation (2.2) still holds in the presence of arbitrary external magnetic fields. Furthermore, we have, to first order in the magnetic field H_n,

$$\tanh(\beta h_n) = \tanh(\beta J(\sigma_{n-1} + \sigma_{n+1})) + \beta H_n(1 - \tanh^2(\beta J(\sigma_{n-1} + \sigma_{n+1}))) + \cdots \tag{5.5}$$

together with the identities

$$\tanh(\beta J(\sigma_{n-1} + \sigma_{n+1})) = \frac{\gamma}{2}(\sigma_{n-1} + \sigma_{n+1})$$
$$\tanh^2(\beta J(\sigma_{n-1} + \sigma_{n+1})) = \frac{\gamma^2}{2}(1 + \sigma_{n-1}\sigma_{n+1}). \tag{5.6}$$

Inserting these expressions into equation (2.2), we readily obtain that, again to first order in the magnetic field $H_n(t)$, the magnetizations $M_n(t)$ obey inhomogeneous differential equations of the form

$$\frac{\mathrm{d}M_n(t)}{\mathrm{d}t} = -M_n(t) + \frac{\gamma}{2}(M_{n-1}(t) + M_{n+1}(t)) + \beta H_n(t) \left(1 - \frac{\gamma^2}{2}(1 + C_{n-1,n+1}(t)) \right) + \cdots. \tag{5.7}$$

As a consequence, the two-time response function $R_n(t, s)$ itself obeys coupled linear differential equations, of the form

$$\frac{\partial R_n(t, s)}{\partial t} = -R_n(t, s) + \frac{\gamma}{2}(R_{n-1}(t, s) + R_{n+1}(t, s)) \tag{5.8}$$

for $t > s$, with the initial value

$$R_n(s, s) = w(s)\delta_{n,0} \tag{5.9}$$

and with

$$w(s) = 1 - \frac{\gamma^2}{2}(1 + C_2(s)) \tag{5.10}$$

where $C_2(s)$ is the equal-time correlation two sites apart, which is given in Laplace space by equation (3.11). Equation (5.8), with its initial condition (5.9), is formally identical to equation (4.3) with initial condition (4.4). Hence its solution reads

$$R_n(s + \tau, s) = w(s)G_n(\tau) \tag{5.11}$$

or, in Laplace space with respect to s,

$$R_n^{L_s}(\tau, p_s) = w^{L_s}(p_s)G_n(\tau). \tag{5.12}$$

Using equation (3.11), we obtain

$$w^{L_s}(p_s) = \frac{p_s + 2}{4p_s}\sqrt{(p_s + 2)^2 - 4\gamma^2} - 1 - \frac{p_s}{4}. \tag{5.13}$$

Let us now discuss the general expression of the response function, given in Laplace space by equations (5.12) and (5.13), along the same lines as for the two-time correlation function.

Equilibrium.   At equilibrium, i.e., for $s \to \infty$ with τ fixed, one gets, either using equation (3.12) or taking the $p_s \to 0$ limit of equation (5.13),

$$w_{\mathrm{eq}} = \sqrt{1 - \gamma^2} \qquad (5.14)$$

hence

$$R_{n,\mathrm{eq}}(\tau) = \sqrt{1 - \gamma^2} G_n(\tau). \qquad (5.15)$$

Comparing this equation to (4.11) leads to the identity

$$R_{n,\mathrm{eq}}(\tau) = -\frac{\mathrm{d}C_{n,\mathrm{eq}}(\tau)}{\mathrm{d}\tau} \qquad (5.16)$$

which is the fluctuation–dissipation theorem (1.5), in dimensionless form.

Non-equilibrium.   For simplicity we again focus our attention on the diagonal component $R_0(t, s)$. In the scaling regime (4.14), we have

$$w^{\mathrm{L}_s}(p_s) = \frac{\sqrt{p_s + \mu^2}}{p_s}. \qquad (5.17)$$

Equation (A.2) implies that in this regime the function $w(s)$ coincides with the amplitude $A(s)$, defined in equation (3.22). Using equation (5.12), and the scaling form (2.20) of $G_0(\tau)$, we obtain the remarkably simple result

$$R_0(s + \tau, s) \approx \frac{A(s)\mathrm{e}^{-\mu^2\tau/2}}{\sqrt{2\pi\tau}}. \qquad (5.18)$$

This general scaling form of the two-time response function further simplifies in the following limiting cases.

(i) At zero temperature, and more generally in the aging regime (1.13), the result (5.18) becomes

$$R_0(s + \tau, s) \approx \frac{1}{\pi\sqrt{2s\tau}}. \qquad (5.19)$$

We thus obtain a scaling form similar to equation (4.28), namely

$$R_0(s + \tau, s) \approx \frac{F_2(x)}{s} \qquad (5.20)$$

with

$$F_2(x) = \frac{1}{\pi\sqrt{2(x - 1)}}. \qquad (5.21)$$

(ii) In the early-time regime (1.14), we obtain

$$R_0(s + \tau, s) \approx \frac{A(s)}{\sqrt{2\pi\tau}} \qquad (5.22)$$

where the right-hand side is identical to equation (4.30).

6. Fluctuation–dissipation ratio

As mentioned in the introduction, a way of characterizing the violation of the fluctuation–dissipation theorem (1.5) in a non-equilibrium situation consists of introducing the fluctuation–dissipation ratio $X(t, s)$ of equation (1.6). In the present situation, we set

$$X(t, s) = \frac{R_0(t, s)}{\partial C_0(t, s)/\partial s} \qquad (6.1)$$

because the response function $R_0(t, s)$ is dimensionless.

In the non-equilibrium scaling regime (4.14), equations (4.23) and (5.18) yield

$$X(s + \tau, s) \approx \frac{e^{-\mu^2 s} + \mu \sqrt{\pi s}\, \text{erf}(\mu \sqrt{s})}{[2(s + \tau)/(2s + \tau)]e^{-\mu^2 s} + \mu \sqrt{\pi s}\, \text{erf}(\mu \sqrt{s})} \qquad (6.2)$$

which is the scaling form of the fluctuation–dissipation ratio in the variables s/τ_{eq} and τ/τ_{eq}. Again we discuss some limiting cases.

 (i) For $s \gg \tau_{\text{eq}}$, with τ and τ_{eq} fixed, the fluctuation–dissipation ratio converges exponentially quickly to its equilibrium value $X_{\text{eq}} = 1$, according to

$$X(s + \tau, s) \approx 1 - \frac{\tau}{2s} \sqrt{\frac{\tau_{\text{eq}}}{2\pi s}} e^{-2s/\tau_{\text{eq}}}. \qquad (6.3)$$

 (ii) The result (6.2) also shows that we have $X(t, s) \approx 1$ in the early-time regime (1.14), consistently with the identity between expressions (4.30) and (5.22). Notice, however, that the initial value $X(s, s)$ of the fluctuation–dissipation ratio is not identically equal to 1. For instance, at zero temperature, equations (4.9) and (5.13) yield after some algebra

$$X(s, s) = 1 - \frac{I_1(2s)}{2s(I_0(2s) + I_1(2s))}. \qquad (6.4)$$

This expression behaves as $X(s, s) \approx \frac{1}{2}$ in the regime $s \ll 1$ of no physical interest, while it converges to unity after a microscopic transient regime, as $X(s, s) \approx 1 - 1/(4s)$ for $s \gg 1$.

(iii) At zero temperature, and more generally in the aging regime (1.13), equation (6.2) simplifies to

$$X(s + \tau, s) \approx \mathcal{X}(x) \qquad (6.5)$$

where the scaling function $\mathcal{X}(x)$ reads

$$\mathcal{X}(x) = \frac{x + 1}{2x}. \qquad (6.6)$$

This result is consistent with the scaling laws (4.28) and (5.20), as we have

$$\mathcal{X}(x) = \frac{F_2(x)}{F_1(x)}. \qquad (6.7)$$

The fluctuation–dissipation ratio decreases from the initial value $\mathcal{X}(0) = 1$, corresponding to equilibrium, to the non-trivial asymptotic value

$$X_\infty = \mathcal{X}(\infty) = \tfrac{1}{2}. \qquad (6.8)$$

Recent developments on aging systems [4] suggest the following alternative presentations of the above results concerning the aging regime.

Firstly, there is a functional relationship $X(C)$ between the fluctuation–dissipation ratio $X \equiv X(t, s)$ and the two-time correlation function $C \equiv C_0(t, s)$ in the aging regime, as
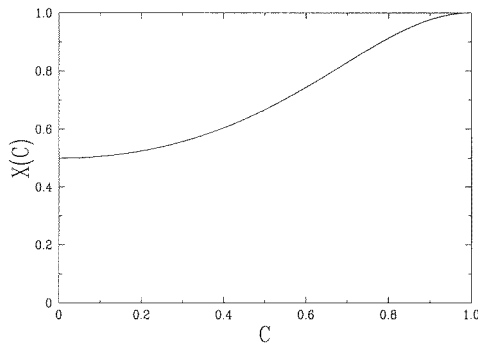
Figure 1. Plot of the relationship $X(C)$ of equation (6.9) between the fluctuation–dissipation ratio $X(t, s)$ and the correlation function $C_0(t, s)$ in the aging regime.
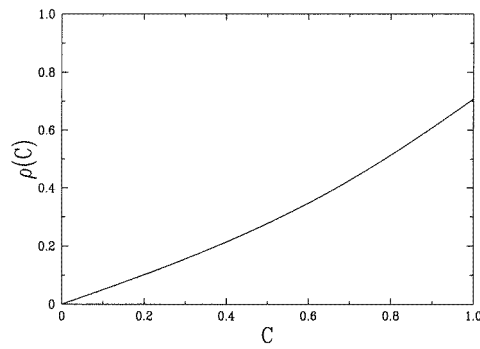
Figure 2. Plot of the relationship $\rho(C)$ of equation (6.14) between the integrated response $\rho_0(t, s)$ and the correlation function $C_0(t, s)$ in the aging regime.

a consequence of equations (4.25) and (6.5). Indeed, eliminating the time ratio x between equations (4.27) and (6.6), we obtain

$$X(C) = \frac{1}{1 + \cos^2(\pi C/2)}. \tag{6.9}$$

Secondly, the dimensionless integrated response function

$$\rho_0(t, s) = \int_0^s R_0(t, u) \, \mathrm{d}u \tag{6.10}$$

proportional to the thermoremanent magnetization [4], reads

$$\rho_0(t, s) = \int_{C_0(t,0)}^{C_0(t,s)} X(t, u) \, \mathrm{d}C_0(t, u) \tag{6.11}$$

using definition (6.1) of $X(t, s)$. In the aging regime, the existence of the functional relationship (6.9) implies

$$\rho_0(t, s) \approx \rho(C_0(t, s)) \tag{6.12}$$

with

$$\rho(C) = \int_0^C X(C') \, \mathrm{d}C' \tag{6.13}$$

i.e., explicitly,

$$\rho(C) = \frac{\sqrt{2}}{\pi} \arctan\left(\frac{1}{\sqrt{2}} \tan\left(\frac{\pi C}{2}\right)\right). \tag{6.14}$$
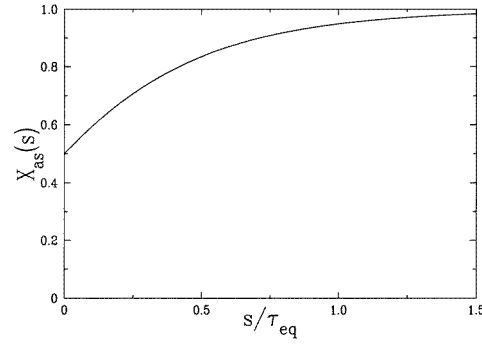
The functions $X(C)$ and $\rho(C)$ are shown, respectively, in figures 1 and 2. For $C \to 1$, i.e. $x = t/s \to 1$, we have

$$X(C) = 1 - \frac{\pi^2}{4}(1 - C)^2 + \cdots \qquad \rho(C) = \frac{1}{\sqrt{2}} - (1 - C) + \frac{\pi^2}{12}(1 - C)^3 + \cdots \tag{6.15}$$

while for $C \to 0$, i.e., $x \to \infty$, we have

$$X(C) = \frac{1}{2} + \frac{\pi^2 C^2}{16} + \cdots \qquad \rho(C) = \frac{C}{2} + \frac{\pi^2 C^3}{48} + \cdots. \tag{6.16}$$

Let us point out that the slope of the $\rho(C)$ curve near the origin is given by the asymptotic value $X(C = 0) = X_\infty = \frac{1}{2}$.

Figure 3. Plot of the asymptotic fluctuation–dissipation ratio $X_{as}(s)$ in the non-equilibrium scaling regime, as given by equation (6.17), against s/τ_{eq}.

It is clear from equation (6.2) that the fluctuation–dissipation theorem is maximally violated for $\tau \gg s$, at least in the scaling regime (2.18). In this situation, equation (6.2) yields the prediction

$$X_{as}(s) \approx \frac{e^{-\mu^2 s} + \mu\sqrt{\pi s}\,\text{erf}(\mu\sqrt{s})}{2e^{-\mu^2 s} + \mu\sqrt{\pi s}\,\text{erf}(\mu\sqrt{s})} \tag{6.17}$$

for the asymptotic fluctuation–dissipation ratio

$$X_{as}(s) = \lim_{\tau \to \infty} X(s + \tau, s). \tag{6.18}$$

Figure 3 shows a plot of the prediction (6.17) for $X_{as}(s)$, against the ratio s/τ_{eq}. In the aging regime ($1 \ll s \ll \tau_{eq}$), the asymptotic fluctuation–dissipation ratio smoothly departs from its limit value $X_\infty = \frac{1}{2}$, as

$$X_{as}(s) \approx \frac{1}{2} + \frac{s}{\tau_{eq}} + \cdots \tag{6.19}$$

while it converges exponentially quickly to its equilibrium value $X_{eq} = 1$ for $s \gg \tau_{eq}$:

$$X_{as}(s) \approx 1 - \sqrt{\frac{\tau_{eq}}{2\pi s}} e^{-2s/\tau_{eq}}. \tag{6.20}$$

7. Discussion

This work is devoted to the non-equilibrium dynamics of the ferromagnetic Ising chain, quenched from infinite temperature to finite temperature, and evolving under Glauber dynamics. We have exploited the solvability of this model in order to derive exact expressions for the spin autocorrelation function $C_0(t, s) = \langle \sigma_0(t)\sigma_0(s) \rangle$, and the associated response function $R_0(t, s)$ and fluctuation–dissipation ratio $X(t, s)$, with t (observation time) $> s$ (preparation time) and $\tau = t - s$. While our study of correlations complements a well-investigated field, the results concerning the response and the fluctuation–dissipation ratio are entirely novel.

We have given a detailed analysis of the scaling regime of low temperatures, large distances (proportional to the equilibrium correlation length ξ_{eq}) and large times (proportional to the relaxation time $\tau_{eq} \sim \xi_{eq}^2$). This scaling regime encompasses several limiting situations of interest:

- Self-similar domain-growth or aging regime ($1 \ll s \sim \tau \ll \tau_{\mathrm{eq}}$). At zero temperature, because of the self-similarity of the coarsening phenomenon, the various two-time observables obey simple scaling laws only involving the time ratio $x = t/s$. This situation is typical of an aging system [4]. The corresponding scaling functions for the autocorrelation function $C_0(t, s)$, its derivative $\partial C_0(t, s)/\partial s$, the response function $R_0(t, s)$, and the associated fluctuation–dissipation ratio $X(t, s)$ are given explicitly in equations (4.27), (4.29), (5.21) and (6.6). These functions are universal, implying, in particular, that they are unchanged if the initial condition contains short-range correlations. These results imply the existence of a non-trivial relationship $X(C)$ (see equation (6.9)) throughout the aging regime. At any low but finite temperature, the self-similar domain growth and the associated aging phenomena are interrupted for times of order τ_{eq}.

- Spatial ($1 \ll |n| \ll \xi_{\mathrm{eq}} \sim \sqrt{s}$) and temporal ($\tau \ll s \sim \tau_{\mathrm{eq}}$) Porod regimes. In these regimes, the autocorrelation function departs from its value of unity in a singular fashion, involving either an $|n|$ [3] or a $\sqrt{\tau}$ dependence (see equations (3.21), (4.31)). We have derived a prediction (3.22) for the common prefactor $A(t)$ of both these laws, as well as the interpolation formula (4.34) in the spatio-temporal regime.

- Equilibrium regime ($\tau \sim \tau_{\mathrm{eq}} \ll s$). At low but finite temperature, the system converges toward equilibrium for times larger than τ_{eq}. The exponential law of convergence has also been studied in detail (see equations (3.19), (6.3), (6.20)).

As already underlined in the introduction, one of the most salient outcomes of this study is the non-trivial limit value $X_\infty = \frac{1}{2}$ of the fluctuation–dissipation ratio for $1 \ll s \ll t$ in the aging regime. The value of X_∞ for a system quenched to its critical point could be *a priori* any number between $X_\infty = 1$ ($T > T_c$: equilibrium) and $X_\infty = 0$ ($T < T_c$: domain growth). It turns out that the answer is exactly half-way between these bounds for the Glauber–Ising chain. A forthcoming work [17] is devoted to the limit fluctuation–dissipation ratio in systems having a finite T_c, and exhibiting domain growth in a whole low-temperature phase. The spherical model in any dimension $d > 2$ and the two-dimensional Ising model will serve as benchmarks and lead us to claim that X_∞ is a novel universal characteristic of critical dynamics, intrinsically related to non-equilibrium phenomena, which can take any value, at least in the range $0 \leqslant X_\infty \leqslant \frac{1}{2}$.

Let us anticipate that the limit fluctuation–dissipation ratio X_∞ appears as an amplitude ratio [17], in the sense used in critical phenomena. For a critical quench in such generic models, we have indeed

$$
\begin{aligned}
C(t, s) &\approx s^{-(d-2+\eta)/z} F(x) \\
\frac{\partial C(t, s)}{\partial s} &\approx s^{-1-(d-2+\eta)/z} F_1(x) \\
T_c R(t, s) &\approx s^{-1-(d-2+\eta)/z} F_2(x).
\end{aligned}
\tag{7.1}
$$

These scaling laws generalize equations (4.25), (4.28), (5.20). The dynamical exponent z is accessible from the study of dynamical critical phenomena at equilibrium, while $x = t/s$ is again the time ratio. The fluctuation–dissipation ratio therefore scales as

$$
X(t, s) \approx \mathcal{X}(x) = \frac{F_2(x)}{F_1(x)}
\tag{7.2}
$$

where the scaling function $\mathcal{X}(x)$ is again universal. For large values of this ratio ($x \gg 1$, i.e.,

$1 \ll s \ll t$), the scaling functions entering equations (7.1) fall off as

$$\begin{aligned} F(x) &\approx B x^{-\lambda_c/z} \\ F_1(x) &\approx B_1 x^{-\lambda_c/z} \\ F_2(x) &\approx B_2 x^{-\lambda_c/z} \end{aligned} \tag{7.3}$$

where λ_c, related to the critical initial-slip exponent θ_c [20] by $\lambda_c = d - z\theta_c$, is a dynamical critical exponent which only manifests itself in non-equilibrium phenomena [21]. The exact results derived in this paper for the Ising chain agree with the above scaling laws for $\eta = 1$, $z = 2$, $\lambda_c = 1$ and $\theta_c = 0$. The limit fluctuation–dissipation ratio

$$X_\infty = \mathcal{X}(\infty) = \frac{B_2}{B_1} \tag{7.4}$$

thus appears as a dimensionless amplitude ratio associated with the behaviour (7.3). It is, therefore, a novel universal quantity in non-equilibrium critical dynamics.

Appendix A. Some useful inverse Laplace transforms

In this appendix we list a few inverse Laplace transforms, which can be found in [22], and have been used in the main body of this paper. Notations are as in equation (2.5). The symbols a, b denote complex parameters with a positive real part.

$f^{\mathrm{L}}(p)$	$f(t)$	
$\dfrac{2\mathrm{e}^{-b\sqrt{p+a^2}}}{p}$	$\mathrm{e}^{ab}\,\mathrm{erfc}\left(\dfrac{b}{2\sqrt{t}}+a\sqrt{t}\right)+\mathrm{e}^{-ab}\,\mathrm{erfc}\left(\dfrac{b}{2\sqrt{t}}-a\sqrt{t}\right)$	(A.1)
$\dfrac{\sqrt{p+a^2}}{p}$	$\dfrac{\mathrm{e}^{-a^2 t}}{\sqrt{\pi t}}+a\,\mathrm{erf}(a\sqrt{t})$	(A.2)
$\dfrac{a}{p\sqrt{p+a^2}}$	$\mathrm{erf}(a\sqrt{t})$	(A.3)
$\dfrac{1}{\sqrt{p+a^2}(b+\sqrt{p+a^2})}$	$\mathrm{e}^{(b^2-a^2)t}\,\mathrm{erfc}(b\sqrt{t})$	(A.4)
$\mathrm{e}^{ap}\,\mathrm{erfc}(\sqrt{ap})$	$\dfrac{1}{\pi(a+t)}\sqrt{\dfrac{a}{t}}.$	(A.5)

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